

Some Rigidity Conditions on Berwald Structures

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This work is dedicated to my parents, Luis y Elvira and to my brothers Elvira Julio and Luis Miguel.

The present manuscript corresponds to the memory of my research work in order to obtain the Masters Degree in Mathematics. It has been written under the supervision of Prof. Fernando Etayo Gordejuela. I am pleased to thank to him for his time, his kind hospitality, his collaboration during the elaboration of the present and related works. Undoubtable, this thesis can not be done without his guidance. Therefore, my gratitude to him.

Sobre una condicion de rigidez de los espacios de Berwald

Resumen. Esta tesis contiene una introducción al método de los promediados de estructuras geometricas, en particular de estructuras definidas por conexiones Finslerianas. Se aplica el método a espacios de Berwald, que son espacios de Finsler pero que preservan todavia mucha de las características propias de los espacios Riemannianos. En este sentido, se obtienen condiciones de rigidez geodésica, como el *teorema* 5.1.3. En la prueba, es esencial el promediado de la conexión the Chern. Mas tarde se muestra que la conexión de Levi Civita de cualquier métrica Riemanniana afínmente equivalente a una estructura de Berwald deja invariante por transporte paralelo la indicatriz de dicha estructura de Berwald. También se demuestra el resultado recíproco: Si (\mathbf{M}, F) es una estructura de Finsler y existe una estructura Riemanniana cuya conexión de Levi Civita deja invariante por transporte paralelo la indicatriz de la estructura de Finsler, entonces (\mathbf{M}, F) es de Berwald. Como aplicación se obtiene una condición necesaria para que una variedad sea de Landsberg pura.

Chapter 1

Introduction

Finsler geometry has its conceptual genesis in the seminar *lecture* of Bernhard Riemann "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" (Habilitationsschrift, 1854, Abhandlungen der Kniglichen Gesellschaft der Wissenschaften zu Gttingen, 13 (1868)). In this work, Riemann introduced the basic ingredients of the modern notion of manifold, Riemannian structure and Finsler structure. However, in the same work he noticed the complication of the (general) Finsler case compared with the Riemannian (quadratic case).

Due to the pre-eminence of the quadratic case, Finsler Geometry was dormant for decades, reappearing in the thesis of P. Finsler (under the supervision of Caratheodory) in 1918. This is the reason why this type of geometry is also known by *Riemann–Finsler* geometry or Finsler Geometry for short.

After Finsler's thesis, an explosion in the field came in the next decades and diverse schools of Finsler Geometry emerged, as well as significant contribution of many geometers. During this earlier development, the results are mainly local in character and related with analytical questions, in particular, the calculus of variations.

One of the relevant figures working on that period on Finsler Geometry was L. Berwald, who introduced a connection and a class of spaces sharing his name. Berwald connection is important because it can be extracted directly from the differential equation stipulated as being the geodesic equations. Berwald spaces are interesting because they are closely related to Riemannian spaces. Berwald spaces are the category that, because being quite close to the Riemannian category, more rigidity conditions can be found. Indeed, a rigidity result due to Szabó says that a Berwald space of dimension 2

is either a Riemannian space or a locally Minkowski space ([6]); therefore to find examples of Berwald spaces it is necessary to go higher dimensions ([1, *chapter 11*]). In addition, Berwald spaces have the interesting feature that they are related to the *Equivalence Principle* of General Relativity: the Berwald connection of a Berwald spaces constitute a general type of torsion-free connections compatible with it.

It is in the category of Berwald Spaces where the present thesis has to be considered. This memory explains in a (hopefully) self-contained way some of the results presented in reference [15] in a jointly work with Prof. Fernando Etayo.

Between the amount of results presented, we would like to mention the following:

1. A result on geodesic rigidity in the category of Berwald spaces (*proposition 5.1.7*), which is similar to a result obtained by V. Matveev ([9])
2. A rigidity result on Berwald spaces, (*proposition 5.2.4*).
3. A rigidity condition for Landsberg spaces *proposition 5.3.2*.

The technical tool used to obtain these results is to consider the average of some geometric Finslerian quantities and in particular, the average of some Finsler linear connection and the averaged of the fundamental tensor. The averaging operation is presented in the pre-print [2] as well as in sub-sequent works. We think that the averaged founded in that reference is useful (for some purposes more than the averaged of the fundamental tensor) because the relation between the average of the curvature of the original connection and the curvature of the averaged connection. This makes this average more powerful than the average of the fundamental tensor.

Chapter 2

Basic Notions on Riemann-Finsler Geometry

Let (x, \mathbf{U}) be a local coordinate system on \mathbf{M} , where $x \in \mathbf{U}$ have local coordinates (x^1, \dots, x^n) , $\mathbf{U} \subset \mathbf{M}$ is an open set and \mathbf{TM} is the tangent bundle of the manifold \mathbf{M} . A tangent vector at the point $x \in \mathbf{M}$ is denoted by $y^i \frac{\partial}{\partial x^i} \in \mathbf{T}_x \mathbf{M}$, $y^i \in \mathbf{R}$. We use Einstein's convention for up and down equal indices in this work if the contrary is not directly stated. We can identify the point x with its coordinates (x^1, \dots, x^n) and the tangent vector $y \in \mathbf{T}_x \mathbf{M}$ at x with its components $y = (y^1, \dots, y^n)$. Then each local coordinate system (x, \mathbf{U}) on the manifold \mathbf{M} induces a local coordinate system on \mathbf{TM} denoted by (x, y, \mathbf{U}) such that $y = y^i \frac{\partial}{\partial x^i} \in \mathbf{T}_x \mathbf{M}$ has local natural coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ in the induced natural coordinate system. The slit tangent bundle is $\pi : \mathbf{N} \longrightarrow \mathbf{M}$ such that $\mathbf{N} = \mathbf{TM} \setminus \{0\}$; i.e., the tangent bundle with the zero section removed.

2.1 Definition of Finsler Structure

Definition 2.1.1 *A Finsler structure F on the manifold \mathbf{M} is a non-negative, real function $F : \mathbf{TM} \rightarrow [0, \infty[$ such that*

1. *It is smooth in the slit tangent bundle \mathbf{N} .*
2. *Positive homogeneity holds: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$.*
3. *Strong convexity holds: the Hessian matrix*

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \quad (2.1.1)$$

is positive definite on \mathbf{N} .

We also denote by a Finsler structure to the pair (\mathbf{M}, F) . The coefficients $g_{ij}(x, y)$ are the components of the fundamental tensor g defined later.

Remark 1. When the second Bianchi identities are used, the minimal smoothness requirement for the Finsler structure is to be \mathcal{C}^5 ; more generally, only \mathcal{C}^4 differentiable structure is required, if one speaks only of curvatures.

Remark 2. The homogeneity condition can be stronger: $F(x, \lambda y) = |\lambda|F(x, y)$. In this case (\mathbf{M}, F) is called absolutely homogeneous Finsler structure.

Remark 3. In some examples it is convenient to reduce the condition of strong convexity in the whole $\mathbf{TM} \setminus \{0\}$ to some proper sub-manifold of \mathbf{N} defined by proper subsets of the tangent spaces. Then one speaks of y -locality of the strong convexity condition.

Definition 2.1.2 ([1]) *Let (\mathbf{M}, F) be a Finsler structure and (x, y, \mathbf{U}) a local coordinate system induced on \mathbf{TM} from the coordinate system (x, \mathbf{U}) of \mathbf{M} . The components of the Cartan tensor are defined by the set of functions:*

$$A_{ijk} = \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad i, j, k = 1, \dots, n. \quad (2.1.2)$$

These coefficients are homogeneous of degree zero in (y^1, \dots, y^n) . In the Riemannian case the coefficients A_{ijk} are zero, and this fact characterizes Riemannian geometry from other types of Finsler geometries (this result is known as Deicke's theorem ([1, pg 393])).

Let us consider the vector bundle $\pi^*\mathbf{TM}$, pull-back bundle of \mathbf{TM} by the projection π , defined as the minimal sub-bundle of the cartesian product $\mathbf{N} \times \mathbf{TM}$ such that the following sub-bundle commutes:

$$\begin{array}{ccc} \pi^*\mathbf{TM} & \xrightarrow{\pi_2} & \mathbf{TM} \\ \pi_1 \downarrow & & \downarrow \pi \\ \mathbf{N} & \xrightarrow{\pi} & \mathbf{M} \end{array}$$

where the projection π is

$$\pi : \mathbf{N} \longrightarrow \mathbf{M}$$

$$u \longrightarrow x, u \in \mathbf{N}, x \in \mathbf{M}.$$

$\pi^*\mathbf{TM}$ has base manifold \mathbf{N} , the fiber over the point $u \in \mathbf{N}$ with coordinates (x, y) is isomorphic to $\mathbf{T}_x\mathbf{M}$ for every point $u \in \pi^{-1}(x)$ and the structure group is isomorphic to $\mathbf{GL}(n, \mathbf{R})$. Given a vector field $Z \in \Gamma\mathbf{TM}$, the corresponding element on the pull-back bundle is defined on each $u \in \pi^{-1}(x) \subset \mathbf{N}$ by the cartesian pair $(u, Z(x))$.

An alternative treatment of Finsler geometry uses the homogeneity properties on y of the different geometric objects that appear in the theory. In fact, for positive homogeneous metrics, one can investigate the geometry of analogous pull-back bundles but where the base manifold is the sphere bundle \mathbf{SM} (or the projective sphere bundle \mathbf{PTM} in the case of absolutely homogeneous structures). The sphere bundle \mathbf{SM} is defined as follows. Consider the manifold \mathbf{N} and the equivalence relation defined as

$$(x, y) \equiv (x, \tilde{y}) \text{ iff } \exists \lambda \in \mathbf{R}^+ \text{ such that } y = \lambda \tilde{y}.$$

Then \mathbf{SM} is a fiber bundle over the manifold \mathbf{M} , with fiber over the point $x \in \mathbf{M}$

$$\pi : \mathbf{SM} \longrightarrow \mathbf{M}, \quad (x, [y]) \longrightarrow x,$$

where $(x, [y])$ is the equivalence class defined by above equivalence relation. Then, one can construct as before the pull-back bundle $\pi^*\mathbf{TM}$. If the structure F is absolutely homogeneous of degree zero, then one can define the projective bundle in a similar way:

$$\pi_S : \mathbf{N} \longrightarrow \mathbf{PTM}, \quad (x, y) \longrightarrow (x, [y]),$$

defining the equivalence class as $[y] := \{(x, y) \mid y = \lambda y_0, \forall \lambda \neq 0\}$.

For example, the matrix coefficients $(g_{ij}(x, y))$ are also invariant under a positive scaling of y and therefore they live on \mathbf{SM} . The Cartan tensor components A_{ijk} also live on \mathbf{SM} , if they are defined according to the formula (2.1.2). If F is absolutely homogeneous rather than positive homogeneous the coefficients g_{ij} and A_{ijk} live on \mathbf{PTM} , the projective tangent bundle. \mathbf{PTM} is defined in a similar way as \mathbf{SM} but the projection also sends y and $-y$ to the same equivalence class $[y_0]$.

Examples of Finsler Structures

1. **Minkowski Space.** Given a vector space \mathbf{V} a Minkowski norm is a map $\|\cdot\| : \mathbf{V} \longrightarrow \mathbf{R}$, such that

- (a) It is non-negative and $\|y\| = 0$ iff $y = 0$.
- (b) It is positive homogeneous of degree 1.
- (c) It is smooth on y and the Hessian respect to y is strictly positive.

A Minkowski space is a pair (\mathbf{V}, F) as above. Indeed, one can check that an ordinary norm is also defined from the axioms of Minkowski norm.

2. **Riemannian Structures.** In this case F has the form $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$, for each $x \in \mathbf{M}$ and $y \in \mathbf{T}_x \mathbf{M}$ and the matrix $g_{ij}(x)$ defines a positive definite, symmetric bilinear form on $\mathbf{T}_x \mathbf{M}$.
3. **(α, β) -metrics.** They are Finsler structures determined by a Riemannian norm $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ and a linear form $\beta := \beta_i(x)y^i$. One of the most interesting cases are Randers structure ([1, chapter 11]), which has the form $F(x, y) = \alpha(x, y) + \beta(x, y)$. The 1-form β has norm less than 1 by the Riemannian norm α . This ensures positivity as well as strong convexity.
4. **Numata metrics.** They are defined by functions of the form $F(x, y) = \alpha(x, y) + \beta(x, y)$, where $\alpha = \sqrt{g_{ij}(y)y^i y^j}$ is a homogeneous function of degree 1.
5. It was proved that the function measuring the time spent climbing a mountain can be represented by a Finsler function. One nice reference is [16]. Let us consider the Finslerian distance between two arbitrary points p and d , which is the infimum of the Finslerian length of all possible piecewise smooth path connecting them:

$$d(p, q) := \inf_{\sigma} \left\{ \int_{\sigma} ds \sqrt{\eta_{ij} \dot{\sigma}^i \dot{\sigma}^j}, \sigma : [0, 1] \longrightarrow \mathbf{M} \right\}, \quad (2.1.3)$$

where \mathbf{M} is a smooth representation of the mountain, η is the inner Riemannian metric on \mathbf{M} induced from the Euclidean metric in \mathbf{R}^3 , σ is a path connecting p and q and $\dot{\sigma}$ the tangent vector along σ . Given a point over a possible path $\sigma(s)$, let us denote the maximal speed as $c(\sigma(s), \dot{\sigma}(s))$. Then, the minimal time is given by:

$$t_{min}(p, q) := \inf_{\sigma} \left\{ \int_{\sigma} ds \sqrt{\frac{\eta_{ij} \dot{\sigma}^i \dot{\sigma}^j}{c^2(\sigma(s), \dot{\sigma}(s))}}, \sigma : [0, 1] \longrightarrow \mathbf{M} \right\}. \quad (2.1.4)$$

Since by definition $c(\sigma(s), \dot{\sigma}(s))$ is homogeneous of degree zero on the second argument, the function

$$F : \mathbf{N} \longrightarrow \mathbf{R}$$

$$(x, y) \longrightarrow \sqrt{\frac{\eta_{ij}y^i y^j}{c^2(x, y)}}. \quad (2.1.5)$$

This is a Finsler metric.

2.2 The Non-Linear connection

An Ehresmann connection in a principal fiber bundle $\pi : \mathbf{P} \longrightarrow \mathbf{M}$ is a splitting of the tangent bundle \mathbf{TP} such that $\mathbf{T}_u \mathbf{P} = \mathcal{V}_u \oplus \mathcal{H}_u$ with $\mathcal{V}_u = \ker d\pi$, for all $u \in \mathbf{P}$.

There is a non-linear connection on the manifold \mathbf{N} . In order to introduce it, let us define the *non-linear connection coefficients*, defined by the formula in local coordinates

$$\frac{N_j^i}{F} = \gamma_{jk}^i \frac{y^k}{F} - A_{jk}^i \gamma_{rs}^k \frac{y^r}{F} \frac{y^s}{F}, \quad i, j, k, r, s = 1, \dots, n$$

where the formal second kind Christoffel symbols γ_{jk}^i are defined by the expression

$$\gamma_{jk}^i = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{sk}}{\partial x^j} \right), \quad i, j, k = 1, \dots, n;$$

$A_{jk}^i = g^{il} A_{ljk}$ and $g^{il} g_{lj} = \delta_j^i$. Note that the coefficients $\frac{N_j^i}{F}$ are invariant under positive scaling $y \rightarrow \lambda y$, $\lambda \in \mathbf{R}^+$, $y \in \mathbf{T}_x \mathbf{M}$.

Let us consider the local coordinate system (x, y, \mathbf{U}) of the manifold \mathbf{TM} . An induced tangent basis for $\mathbf{T}_u \mathbf{N}$, $u \in \mathbf{N}$ is defined by the vectors([2]):

$$\left\{ \frac{\delta}{\delta x^1} |_u, \dots, \frac{\delta}{\delta x^n} |_u, F \frac{\partial}{\partial y^1} |_u, \dots, F \frac{\partial}{\partial y^n} |_u \right\},$$

$$\frac{\delta}{\delta x^j} |_u = \frac{\partial}{\partial x^j} |_u - N_j^i \frac{\partial}{\partial y^i} |_u, \quad i, j = 1, \dots, n. \quad (2.2.1)$$

The local sections $\left\{ \frac{\delta}{\delta x^1} |_u, \dots, \frac{\delta}{\delta x^n} |_u, u \in \pi^{-1}(x), x \in \mathbf{U} \right\}$ generates the local horizontal distribution \mathcal{H}_U , while $\left\{ \frac{\partial}{\partial y^1} |_u, \dots, \frac{\partial}{\partial y^n} |_u, u \in \pi^{-1}(x), x \in \mathbf{U} \right\}$ the

local vertical distribution \mathcal{V}_U . The subspaces \mathcal{V}_u and \mathcal{H}_u are such that the following splitting of $\mathbf{T}_u\mathbf{N}$ holds:

$$\mathbf{T}_u\mathbf{N} = \mathcal{V}_u \oplus \mathcal{H}_u, \forall u \in \mathbf{N}.$$

This decomposition is invariant by the action of $\mathbf{GL}(n, \mathbf{R})$ and it defines a non-linear connection (a connection in the sense of Ehresmann([3])) on the principal fiber bundle $\mathbf{N}(\mathbf{M}, \mathbf{GL}(n, \mathbf{R}))$.

The local basis of the dual vector space $\mathbf{T}_u^*\mathbf{N}$, $u \in \mathbf{N}$ is

$$\begin{aligned} & \{dx^1|_u, \dots, dx^n|_u, \frac{\delta y^1}{F}|_u, \dots, \frac{\delta y^n}{F}|_u\}, \\ & \frac{\delta y^i}{F}|_u = \frac{1}{F}(dy^i + N_j^i dx^j)|_u, \quad i, j = 1, \dots, n. \end{aligned} \quad (2.2.2)$$

This basis is dual to the basis (2.2.1).

2.3 The Chern connection and other connections

The non-linear connection defined above provides the possibility to define an the Chern connection. Let us consider $x \in \mathbf{M}$, $u \in \mathbf{T}_x\mathbf{M} \setminus \{0\}$ and $\xi \in \mathbf{T}_x\mathbf{M}$. We define the canonical projections

$$\begin{aligned} \pi : \mathbf{N} &\longrightarrow \mathbf{M}, & u &\longrightarrow x, \\ \pi_1 : \mathbf{N} \times \mathbf{TM} &\longrightarrow \mathbf{N}, & (u, \xi_x) &\longrightarrow u, \\ \pi_2 : \mathbf{N} \times \mathbf{TM} &\longrightarrow \mathbf{TM}, & (u, \xi_x) &\longrightarrow \xi_x, \end{aligned}$$

with $u \in \pi^{-1}(x)$, $x \in \mathbf{M}$. Then the vector bundle $\pi^*\mathbf{TM}$ is completely determined as the minimal subset of $\mathbf{N} \times \mathbf{TM}$ by the equivalence relation defined in the following way: for every $u \in \mathbf{N}$ and $(u, \xi) \in \pi_1^{-1}(u)$,

$$(u, \xi) \in \pi^*\mathbf{TM} \quad \text{iff} \quad \pi \circ \pi_2(u, \xi) = \pi \circ \pi_1(u, \xi).$$

We can define in a similar way the vector bundle $\pi^*\mathbf{SM}$ over \mathbf{SM} , being $\pi : \mathbf{SM} \longrightarrow \mathbf{M}$ the canonical projection in case of positive homogeneous Finsler structures or $\mathbf{PTM} \longrightarrow \mathbf{M}$ in case of absolutely homogeneous Finsler structures.

Definition 2.3.1 *Let (\mathbf{M}, F) be a Finsler structure. The fundamental and the Cartan tensors are defined in the natural local coordinate system (x, y, \mathbf{U}) by the equations:*

1. *Fundamental tensor:*

$$g(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} dx^i \otimes dx^j. \quad (2.3.1)$$

2. *Cartan tensor:*

$$A(x, y) := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} \frac{\delta y^i}{F} \otimes dx^j \otimes dx^k = A_{ijk} \frac{\delta y^i}{F} \otimes dx^j \otimes dx^k. \quad (2.3.2)$$

The Chern connection ∇ is determined through the structure equations for the connection 1-forms as follows ([1]),

Theorem 2.3.2 *Let (\mathbf{M}, F) be a Finsler structure. The vector bundle $\pi^*\mathbf{TM}$ admits a unique linear connection characterized by the connection 1-forms $\{\omega_j^i, i, j = 1, \dots, n\}$ such that the following structure equations hold:*

1. “Torsion free” condition,

$$d(dx^i) - dx^j \wedge w_j^i = 0, \quad i, j = 1, \dots, n. \quad (2.3.3)$$

2. *Almost g -compatibility condition,*

$$dg_{ij} - g_{kj} w_i^k - g_{ik} w_j^k = 2A_{ijk} \frac{\delta y^k}{F}, \quad i, j, k = 1, \dots, n. \quad (2.3.4)$$

The torsion freeness condition is equivalent to the absence of terms containing dy^i in the connection 1-forms ω_j^i and also implies the symmetry of the connection coefficients Γ_{jk}^i ([2]):

$$w_j^i = \Gamma_{jk}^i dx^k, \quad \Gamma_{jk}^i = \Gamma_{kj}^i. \quad (2.3.5)$$

The torsion freeness condition and almost g -compatibility determines the expression of the connection coefficients of the Chern connection in terms of the Cartan and fundamental tensor components ([1]).

Following theorem (2.3.2), there is a coordinate-free characterization of the Chern connection. Let us denote by $V(\tilde{X})$ the vertical and by $H(\tilde{X})$ the horizontal components (defined by the non-linear connection on \mathbf{N}) of an arbitrary tangent vector $\tilde{X} \in \mathbf{T}_u\mathbf{N}$. Then the following corollaries are immediate consequences of *theorem 2.3.2*:

Corollary 2.3.3 *Let (\mathbf{M}, F) be a Finsler structure. The almost g -compatibility condition (2.3.4) is equivalent to the conditions:*

$$\nabla_{V(\tilde{X})}g = 2A(\tilde{X}, \cdot, \cdot), \quad (2.3.6)$$

$$\nabla_{H(\tilde{X})}g = 0, \forall \tilde{X} \in \mathbf{TN}. \quad (2.3.7)$$

Proof: using local natural coordinates and reading from *theorem 2.3.4*, we have that the covariant derivative of the metric is

$$\nabla(g) = (dg_{ij} - g_{kj}w_i^k - g_{ik}w_j^k)\pi^*e^i \otimes \pi^*e^j = 2A_{ijk}\frac{\delta y^k}{F} \otimes \pi^*e^i \otimes \pi^*e^j.$$

By the definition of covariant derivative along a direction and nothing that $2A_{ijk}\frac{\delta y^k}{F}$ is vertical, one gets,

$$\nabla_{\tilde{X}}(g) := 2A_{ijk}\frac{\delta y^k}{F}(\tilde{X})\pi^*e^i \otimes \pi^*e^j, \forall \tilde{X} \in \mathbf{TN}.$$

From this formula follows the result. \square

Corollary 2.3.4 *Let (\mathbf{M}, F) be a Finsler structure. The torsion-free condition (2.3.3) is equivalent to the following conditions:*

1. *Null vertical covariant derivative of sections of $\pi^*\mathbf{TM}$: let $\tilde{X} \in \mathbf{TN}$ and $Y \in \mathbf{TM}$, then*

$$\nabla_{V(\tilde{X})}\pi^*Y = 0. \quad (2.3.8)$$

2. *Let us consider $X, Y \in \mathbf{TM}$ and the associated horizontal vector fields $\tilde{X} = X^i\frac{\delta}{\delta x^i}$ and $\tilde{Y} = Y^i\frac{\delta}{\delta x^i}$. Then the following equality holds:*

$$\nabla_{\tilde{X}}\pi^*Y - \nabla_{\tilde{Y}}\pi^*X - \pi^*([X, Y]) = 0. \quad (2.3.9)$$

Proof: as before we consider the torsion condition in local coordinates. Then the local frame $\{e_j\}$ of $\Gamma\mathbf{TM}$ commutes, $[e_i, e_j] = 0$. Using the symmetry in the connection coefficients and the definition of the torsion operator (2.11), one obtains that

$$\nabla_{\tilde{e}_i}\pi^*e_j - \nabla_{\tilde{e}_j}\pi^*e_i - \pi^*([e_i, e_j]) = \nabla_{\tilde{e}_i}\pi^*e_j - \nabla_{\tilde{e}_j}\pi^*e_i = (\Gamma_{ij}^k - \Gamma_{ji}^k)\pi^*e_k = 0.$$

In order to proof the second condition, it is as follows,

$$\nabla_{\frac{\partial}{\partial y^i}}\pi^*e_j := \pi^*e_k w_j^k \left(\frac{\partial}{\partial y^i}\right) = \pi^*e_k \Gamma_{dj}^k dx^d \left(\frac{\partial}{\partial y^i}\right) = 0. \square$$

The curvature endomorphisms associated with the connection w are determined by the Cartan's second structure equations,

$$\Omega_j^i := dw_j^i - w_j^k \wedge w_k^i, \quad i, j, k = 1, \dots, n. \quad (2.3.10)$$

In local coordinates, these curvature endomorphisms are decomposed in the following way,

$$\Omega_j^i = \frac{1}{2} R_{jkl}^i dx^k \wedge dx^l + P_{jkl}^i dx^k \wedge \frac{\delta y^l}{F} + \frac{1}{2} Q_{jkl}^i \frac{\delta y^k}{F} \wedge \frac{\delta y^l}{F}. \quad (2.3.11)$$

The quantities R_{jkl}^i , P_{jkl}^i and Q_{jkl}^i are called the hh, hv, and vv-curvature tensor components of the Chern connection. In particular, for the Chern connections it holds that the last component is identically zero $Q = 0$ ([1, chapter 3]) for arbitrary Finsler structures. The other tensors have the following expressions:

$$R_{jkl}^i = \frac{\delta \Gamma_{jk}^i}{\delta x^l} - \frac{\delta \Gamma_{jl}^i}{\delta x^k} - \Gamma_{hl}^i \Gamma_{jk}^h + \Gamma_{hk}^i \Gamma_{jl}^h, \quad (2.3.12)$$

$$P_{jkl}^i = -F \frac{\partial \Gamma_{jk}^i}{\partial y^l}. \quad (2.3.13)$$

Let us mention other examples of linear connections that are relevant in Finsler geometry([1]):

1. is Cartan's connection, which is metric compatible, but has non-trivial torsion. It is determined by the following connection forms:

$$({}^c \omega)_i^k = \omega_i^k + A_{ij}^k \frac{\delta y^j}{F}, \quad i, j, k = 1, \dots, n.$$

2. Berwald's connection, defined by the 1-form connection forms

$$({}^b \omega)_i^k = \omega_i^k + A_{ij}^k dx^j, \quad i, j, k = 1, \dots, n.$$

It is torsion-free, although it is not metric compatible.

Chapter 3

Introduction to Berwald Spaces

We will follow in this *chapter* the corresponding *chapters* 10 and 11 of reference [1]. The proofs of the following statements can be found in this reference.

3.1 Definition and general properties of Berwald Spaces

Definition 3.1.1 *A Finsler structure F is said to be of Berwald type if the coefficients of the Chern connection Γ_{jk}^i , written in natural coordinates, do not depend on y .*

There is a nice characterization of Berwald spaces: (\mathbf{M}, F) is a Berwald space iff the Chern's connection leaves invariant the value of the finsler norm along any curve on \mathbf{M} (see for instance [1], [6] or [9]).

Berwald spaces are slightly different than Riemannian spaces, which are contained in the finsler category. This make them more treatable than other kind of Finsler spaces. Indeed, there is a complete classification of Berwald spaces ([6]). From a physical point of view, Berwald spaces can hold the Equivalence Principle, which lies on the foundations of General Relativity.

A direct consequence of this definition is that for a Berwald structure, the Chern connection defines *per se* a linear connection on the manifold \mathbf{M} .

Therefore, there is defined a covariant derivative on \mathbf{M} :

$$\nabla_X W = \left(\frac{dW^i}{dt} \Big|_{\sigma(t)} + W^j \Gamma_{jk}^i(\sigma(t)) \right) \frac{\partial}{\partial x^i} \Big|_{\sigma(t)}, \quad T = \frac{d\sigma(t)}{dt}.$$

There is the following result ([1]):

Proposition 3.1.2 *Let (\mathbf{M}, F) be a Berwald space. Then:*

1. *Given any parallel vector field W along a curve σ in \mathbf{M} , its Finslerian norm $F(W) = \sqrt{g_W(W, W)}$ is necessarily constant along σ .*
2. *For \mathbf{M} connected, its Minkowski linear spaces $(\mathbf{T}_x \mathbf{M}, F_x)$ are all linearly isometric to each other.*

There are several characterizations of Berwald spaces:

Proposition 3.1.3 *Let (M, F) be a Finsler manifold. Then the following criteria are equivalent:*

1. *The hv-curvature is vanishes: $P_{jkl}^i = 0$.*
2. *The Cartan tensor is covariantly constant along all horizontal directions on the slit tangent bundle $\mathbf{TM} \setminus 0$, $A_{ijk|l}$.*
3. *(\mathbf{M}, F) is a Berwald space.*
4. *$(\Gamma_{jk}^i y^j y^k)_{y^p y^q}$ does not depend on y .*
5. *$(\gamma_{jk}^i y^j y^k)_{y^p y^q}$ does not depend on y .*

The following *proposition* also holds,

Proposition 3.1.4 *Let (\mathbf{M}, F) be a Finsler structure. Then*

1. *The structure is Berwald.*
2. *$P_{jkl}^i = {}^b P_{jkl}^i = 0$, where ${}^b P_{jkl}^i$ is the hv-curvature of the Berwald connection.*
3. *The hh-curvature is given by:*

$$R_{jkl}^i = \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{kl}^i}{\partial x^j} - \Gamma_{hk}^i \Gamma_{jl}^h - \Gamma_{hl}^i \Gamma_{jk}^h.$$

In order to write the following proposition, we need to have the following definition

Definition 3.1.5 *A Finsler structure (\mathbf{M}, F) is called locally Minkowski if at each point $x \in \mathbf{M}$ there is a local coordinate system such that $F(y)$ does not depend on x .*

The locally Minkowski spaces are characterized by

Proposition 3.1.6 *Let (\mathbf{M}, F) be a Finsler manifold. Then the following statements are equivalent,*

1. *Both the R and P curvatures of the Chern connection vanishes.*
2. *The structure is locally Minkowski.*

A Finsler surface is a 2-dimensional surface endowed with a Finsler structure ([1, chapter 4]). In the case of surfaces, the hh -curvature corresponds to the curvature scalar function K , the analogous to the Gaussian curvature for Finsler geometry. Similarly, the hv -curvature is defined by the Cartan invariant I . However, for Berwald surfaces $I = 0$.

In order to introduce the following result, note that the Flag curvature at the point x with flag y and transverse edge V is given by

$$K(x, y) = \frac{V^i y^j R_{ijkl}(x, y) y^l V^k}{g_{(x, y)}(V, V) g_{(x, y)}(y, y) - g_{(x, y)}^2(y, V)},$$

where the evaluation of all the quantities is done at the point (x, y) . One can now state Szabó rigidity theorem:

Theorem 3.1.7 *Let (\mathbf{M}, F) be a connected Berwald surface for the Finsler function F such that is strongly convex in all $\mathbf{TM} \setminus \{0\}$. Then,*

1. *If the curvature $K = 0$, then F is locally Minkowski everywhere.*
2. *If the flag curvature K is not identically zero, then F is Riemannian everywhere.*

This result restricts the existence of pure Berwald spaces (that means, the ones which are not Riemannian or locally Minkowski) to higher dimension than two.

3.2 Examples of Berwald Spaces

Following the end of the *section 3.1*, we give some examples of Berwald spaces.

1. **Riemannian Spaces in dimension n .** They are characterized by the fact that the fundamental tensor g_{ij} defines a quadratic form on \mathbf{M} given by the fundamental tensor. In particular, this implies that This implies that $P_{jkl}^i = 0$, which means that the space is Berwald. Alternatively, one can check that in any natural coordinate system, the connection coefficients of the Chern's connection does not depend on y . Indeed, the connection coefficients of the Chern connection of the Riemannian metric g are equal to the Christoffel symbols of the Levi-Civita connection.
2. **Locally Minkowski Spaces in dimension n .** There is a natural coordinate system where the fundamental tensor is constant on x . Therefore the Christoffel "type" symbol $\gamma_{jk}^i = 0$ as well as the non-linear connection coefficients N_j^i are zero (because they linear combination of the γ functions), in the given natural coordinate system.
3. By Szabó's rigidity theorem ([6]), if we look for a y -global Berwald structures which are not Riemannian or locally Minkowski, one needs to look for in dimensions higher dimensions than 2. However, there are Berwald local surfaces, as the following example due to Berwald and Rund shows ([1, *section 10.3*):

(a) **Example of a y -local Berwald Surface.**

In this example, \mathbf{M} is \mathbf{R}^2 . The Finsler function is given by a function $\xi(x^1, x^2)$ that is a non-constant solution of the PDE

$$\xi \frac{\partial \xi}{\partial x^1} - \xi \frac{\partial \xi}{\partial x^2} = 0.$$

The solutions are given implicitly by ([1] and references there)

$$x^1 + x^2 \xi = \psi(\xi)$$

where ψ is an arbitrary analytic function of xi such that $\psi'' \neq 0$. Finally, the Finsler function is

$$F(x, y) = y^2 \left(\xi + \frac{y^1}{y^2} \right) \quad (3.2.1)$$

$F < 0$ if $y^2 < 0$. Therefore it is a y -local strong convex. The Cartan invariant is $I = 0$, while the sectional curvature is

$$K(x, y) = \frac{\psi''(\xi)}{(\xi + \frac{y^1}{y^2})^3 (\psi'(\xi) - x^2)^3}.$$

Therefore $K \neq 0$. From the form of the function F one notes that the structure is not analytical in the whole $\mathbf{T}_x \mathbf{R}^2$.

(b) **Example of a y -global non-trivial Berwald space.**

In order to give an example of a y -global Berwald structure, we use a Randers metric.

Our example is based on the following result ([1, section 11.6]),

Theorem 3.2.1 *Let (\mathbf{M}, F) be a Randers Space. Denote the underlying Riemannian metric by a , its Levi-Civita connection by γ_{jk}^i and the underlying 1-form by b . Assume*

- i. $\|b\|_a < 1$.*
- ii. The covariant derivative respect the Levi-Civita connection of b vanishes in all directions,*

$$b_{j|k} := \frac{\partial b_j}{\partial x^k} - b_s \gamma_{jk}^s = 0.$$

Then the Randers space is of Berwald type. Conversely, if the Randers space is of Berwald type, then above both conditions hold.

Remark. There is at least one topological restriction to the above construction. The parallel condition is equivalent to the existence of a global non-zero everywhere vector field. Therefore, using Poincare-Hopf index theorem, for compact manifolds without boundary surface the Euler characteristic $\chi \mathbf{M}$ must vanish. The example that we present is the following. The base manifold is given by $\mathbf{M} = \mathbf{S}^2 \times \mathbf{S}^1$. The Riemannian metric is

$$a = (\sin^2(\phi) d\theta \otimes d\theta + d\phi \otimes d\phi) + dt \otimes dt.$$

The parallel 1-form is given by

$$b(x, y) = \epsilon dt, \quad |\epsilon| < 1.$$

In local coordinates a tangent vector $y \in \mathbf{T}_x \mathbf{M}$ can be written as

$$y = y^\theta \partial_\theta + y^\Phi \partial_\Phi + y^t \partial_t.$$

Then, the Finsler function is

$$F(x, y) = \sqrt{\sin^2(\phi)(y^\theta)^2 + (y^\Phi)^2 + (y^t)^2} + \epsilon y^t. \quad (3.2.2)$$

Therefore, by *theorem* (3.2.1) this function F defines a Berwald structure on $\mathbf{S}^2 \times \mathbf{S}^1$. It is clear that this construction can be generalized to higher dimensions, with similar constructions on $\mathbf{S}^n \times \mathbf{S}^1$.

Chapter 4

Review of the Theory of the Averaged Structures Associated with Finsler Structures

This *chapter* follows quite closely *section 4* of [8], where the original theory of averages of geometric structures was presented. As such, this *chapter* does not constitute a original result of the present memory, although it is fundamental for it. However, some of the statements are proved in another way, while some of the proves have been omitted for brevity of this memory.

4.1 The Averaged of Linear Connections

Definition 4.1.1 *Let (\mathbf{M}, F) be a Finsler function. Then the indicatrix at the point $x \in \mathbf{M}$ is the convex sub-manifold $\mathbf{I}_x \subset \mathbf{t}_x\mathbf{M}$ defined by the condition that $(x, y) \in \mathbf{I}_x$ iff $F(x, y) = 1$.*

This is equivalent to the definition of the tangent sphere in Riemannian Geometry. From this perspective, a Finsler structure is a smooth collection $\{\mathbf{I}_x \subset \mathbf{t}_x\mathbf{M}, x \in \mathbf{M}\}$ of smooth, convex tangent sets, one at each point $x \in \mathbf{M}$, while a Riemannian structure is a smooth collection of tangent ellipsoids.

Let \tilde{X} be a tangent vector field along the horizontal path $\tilde{\gamma} : [0, 1] \longrightarrow \mathbf{N}$ connecting the points $u \in \mathbf{I}_x$ and $v \in \mathbf{I}_z$. The parallel transport associated with the Chern connection along $\tilde{\gamma}$ of a section $S \in \pi^*\mathbf{TM}$ is denoted by $\tau_{\tilde{\gamma}}S$.

The parallel transport along $\tilde{\gamma}$ of the point $u \in \mathbf{I}_x$ is $\tau_{\tilde{\gamma}}(u) = \tilde{\gamma}(1) \in \pi^{-1}(z)$. We say that $\tilde{\gamma}$ is horizontal if the tangent vectors along $\tilde{\gamma}$ are horizontal. The horizontal lift of a path is defined using the non-linear connection defined on the bundle $\mathbf{TN} \rightarrow \mathbf{N}$.

Proposition 4.1.2 (*Invariance of the indicatrix by horizontal parallel transport*) Let (\mathbf{M}, F) be a Finsler structure, $\tilde{\gamma} : [0, 1] \rightarrow \mathbf{N}$ the horizontal lift of the path $\gamma : [0, 1] \rightarrow \mathbf{M}$ joining x and z . Then the value of the function $F(x, y)$ is invariant along $\tilde{\gamma}$. In particular, let us consider \mathbf{I}_x (resp \mathbf{I}_z) to be the indicatrix over x (resp z). Then $\tau_{\tilde{\gamma}}(\mathbf{I}_x) = \mathbf{I}_z$.

Proof: Let \tilde{X} be the horizontal lift in \mathbf{TN} of the tangent vector field X along the path $\gamma \subset \mathbf{M}$ joining x and z , $S_1, S_2 \in \pi^*(\mathbf{T}_x\mathbf{M})$. Then *corollary* 2.3.3 implies $\nabla_{\tilde{X}}g(S_1, S_2) = 2A(\tilde{X}, S_1, S_2) = 0$ because the vector field \tilde{X} is horizontal and the Cartan tensor is evaluated in the first argument. Therefore the value of the Finslerian norm $F(x, y) = \sqrt{g_{ij}(x, y)y^i y^j}$, $y \in \mathbf{T}_x\mathbf{M}$, Y with $Y = \pi^*y$ is conserved by horizontal parallel transport,

$$\nabla_{\tilde{X}}(F^2(x, y)) = \nabla_{\tilde{X}}(g(x, y))(Y, Y) + 2g(x, \nabla_{\tilde{X}}Y) = 0,$$

being $\tilde{X} \in \mathbf{TN}$ an horizontal vector. The first term is zero because the above calculation. The second term is zero because of the definition of parallel transport of sections $\nabla_{\tilde{X}}Y = 0$. In particular the indicatrix \mathbf{I}_x is mapped to \mathbf{I}_z because parallel transport is a diffeomorphism. \square

Remark. Note the difference between this statement and the statement of *proposition 3.1.2*: while *proposition 4.1.2* applies to a general Finsler structure, *proposition 3.1.2* refers to Berwald structures, where the Chern's connection defines an affine connection on \mathbf{M} directly. Then, the parallel transport along curves on \mathbf{M} makes sense.

Let us denote by $\pi_v^*\mathbf{\Gamma M}$ the fiber over $v \in \mathbf{N}$ and by $\mathbf{\Gamma}_x\mathbf{M}$ the space of tensors restricted to $x \in \mathbf{M}$; for every tensor $S_x \in \mathbf{\Gamma}_x\mathbf{M}$ and $v \in \pi^{-1}(z)$, $z \in \mathbf{U} \subset \mathbf{M}$ we consider the homomorphism:

$$\pi_2|_v : \pi_v^*\mathbf{\Gamma M} \rightarrow \mathbf{\Gamma}_z\mathbf{M}, \quad S_v \rightarrow S_z$$

$$\pi_v^* : \mathbf{\Gamma}_z\mathbf{M} \rightarrow \pi_v^*\mathbf{\Gamma M}, \quad S_z \rightarrow \pi_v^*S_z.$$

Let $S \in \Gamma\pi^*\mathbf{TM}$ and $S(u) := S_u \in \pi^{-1}(u)$. Then, it holds that

$$\pi_u^* \pi_2|_v S(v) = S(u), \quad u, v \in \pi^{-1}(x). \quad (4.1.1)$$

If $S(v) \in \Gamma_v \pi^* \Gamma M$, $S(u) \in \Gamma_u \pi^* \Gamma M$, the fibers over u and v respectively of the bundle $\pi^* \mathbf{TM} \rightarrow \mathbf{n}$, we have that in a local frame $S(v) = \xi^J(x) \pi_v^* e_J|_x$ and respectively $S(u) = \xi^J(x) \pi_u^* e_J|_x$, where we are using multi-index notation. Therefore,

$$\pi_2|_v S(v) = \pi_2(\xi^J(x) \pi_v^* e_J|_x) = \xi^J(x) e_J|_x.$$

Then,

$$\pi_u^* \pi_2|_v S(v) = \pi_u^* \xi^J(x) e_J|_x = \xi^J(x) \pi_u^* e_J|_x = S(u).$$

Note that for arbitrary $u, v \in \pi^{-1}(x)$ and an arbitrary element $S_u \in \pi_u^* \mathbf{TM}$, in general is not true that

$$\pi_u^* \pi_2|_v : \pi_v^* \Gamma \mathbf{M} \rightarrow \pi_u^* \Gamma \mathbf{M}, \quad S_v \rightarrow S_u$$

because $\pi_u^* \pi_2|_v S_v = \pi_u^* S_x$ and it is not the same than $S_u \in \pi_1^{-1}(u)$, the evaluation of the section $S \in \pi^* \Gamma M$ at the point u .

We are now ready to define the averaging operation,

Definition 4.1.3 Consider the family of automorphisms $A_w := \{A_w : \pi_w^* \mathbf{TM} \rightarrow \pi_w^* \mathbf{TM}\}$ with $w \in \pi^{-1}(x)$ with $x \in \mathbf{M}$. The average of this family of operators is the operator $A_x : \mathbf{T}_x \mathbf{M} \rightarrow \mathbf{T}_x \mathbf{M}$ such that:

$$\langle A_w \rangle := \langle \pi_2|_u A \pi_u^* \rangle_u S_x = \frac{1}{\text{vol}(\mathbf{I}_x)} \left(\int_{\mathbf{I}_x} \pi_2|_u A_u \pi_u^* d\text{vol} \right) S_x,$$

$$\text{vol}(\mathbf{I}_x) = \int_{\mathbf{I}_x} d\text{vol}, \quad u \in \pi^{-1}(x), S_x \in \Gamma_x \mathbf{M}; \quad (4.1.2)$$

$d\text{vol}$ is the standard volume form induced on the indicatrix \mathbf{I}_x from the Riemannian volume of the Riemannian structure $(\mathbf{T}_x \mathbf{M} \setminus \{0\}, g_x)$.

Meaning of the Averaging Operation

This definition of the averaged operation is new, compared with other averages:

1. For instance, in the theory of characteristic classes, integration along the fiber commutes with the exterior differential and this is an essential point to prove Thom's isomorphism theorem ([10]). The integration is in this example of forms on fibers that are finite vectors spaces.

2. In Classical Mechanics, integration along the fiber is used to derive a simplified averaged model, which in some circumstances is simpler to analyze ([11]). This is also an integration along fiber, where the fibers are invariant tori.

In these both cases, the fiber bundle structure is similar: we have a bundle $\pi : \mathbf{P} \longrightarrow \mathbf{M}$ and then we calculate the integrals on $\pi^{-1}(x)$ for a given $x \in \mathbf{M}$.

However, the averaging procedure that we propose is a bit more involved. In our case, we have a double fiber structure:

$$\pi^*\mathbf{M} \xrightarrow{\pi_1} \mathbf{N} \xrightarrow{\pi} \mathbf{M}.$$

Although the composition is also a fiber bundle $\pi_1 \circ \pi : \pi^*\mathbf{TM} \longrightarrow \mathbf{M}$, the integration that we performed is on a lift of the fiber in the intermediate base manifold $\mathbf{I}_x \subset \mathbf{N}$ on $\pi^*\mathbf{TM}$. Therefore we need in this case more structure than an ordinary fiber bundle structure. In particular, we need to fix the lift.

Let $f \in \mathcal{FM}$ be a real, smooth function on the manifold \mathbf{M} ; $\pi^*f \in \pi^*\mathcal{FM}$ is defined in the following way,

Definition 4.1.4 *Let (\mathbf{M}, F) be a Finsler structure, $\pi(u) = x$ and consider $f \in \mathcal{FM}$. Then*

$$\pi_u^*f = f(x), \quad \forall u \in \pi^{-1}(x). \quad (4.1.3)$$

The definition is consistent because the function π_v^*f is constant for every $v \in \pi^{-1}(x)$: $\pi_u^*f = f(x) = \pi_v^*f, \forall u, v \in \pi^{-1}(x)$. Therefore the image is the constant value $f(x)$ for every $w \in \pi^{-1}(x)$. $\pi_u^* : \mathbf{T}_x\mathbf{M} \longrightarrow \pi_u^*\mathbf{TM}$ is an isomorphism between $\mathbf{T}_x\mathbf{M}$ and $\pi_1^{-1}(u)$, $\forall x \in \mathbf{M}, u \in \pi^{-1}(x)$.

Let us denote the horizontal lifting operator in the following way:

$$\iota : \mathbf{TM} \longrightarrow \mathbf{TN}, \quad X = X^i \frac{\partial}{\partial x^i} \Big|_x \longrightarrow \tilde{X} = X^i \frac{\delta}{\delta x^i} \Big|_u = \iota(X), \quad u \in \pi^{-1}(x). \quad (4.1.4)$$

This homomorphism is injective and the final result is a section of the tensor bundle \mathbf{TN} . In addition, it defines unambiguously a horizontal tangent vector $\tilde{X} \in \mathcal{H}_u$ for every tangent vector $X \in \mathbf{T}_x\mathbf{M}$. In our calculations, We will also consider the restrictions of this map $\{\iota_u, u \in \pi^{-1}(x)\}$, such that $\iota_u(X) = (\iota X)_u, X \in \mathbf{T}_x\mathbf{M}$.

The following *proposition* is the basis of the theory of the averaged structures associated with Finsler structures. The original proof can be found in reference [8], as well as all the proofs of the results presented in this section,

Theorem 4.1.5 *Let (\mathbf{M}, F) be a Finsler structure and $u \in \pi^{-1}(x)$, with $x \in \mathbf{M}$ and let us consider the respective Chern connection ∇ . Then for each tangent field $X \in \mathbf{T}_x\mathbf{M}$ there is defined on \mathbf{M} a covariant derivative $\tilde{\nabla}_X$ such that*

1. $\forall X \in \mathbf{T}_x\mathbf{M}$ and $Y \in \mathbf{TM}$ the covariant derivative of Y in the direction X is given by the following average:

$$\tilde{\nabla}_X Y = \langle \pi_2|_u \nabla_{\iota_u(X)} \pi_v^* Y \rangle_u, \quad u \in \mathbf{I}_x \subset \pi^{-1}(x) \subset \mathbf{N}. \quad (4.1.5)$$

2. For every smooth function $f \in \mathcal{FM}$ the covariant derivative is given by the following average:

$$\tilde{\nabla}_X f = \langle \pi_2|_u \nabla_{\iota_u(X)} \pi_v^* f \rangle_u = X \cdot (f). \quad (4.1.6)$$

Proof: The argument follows in the following way. Consider the convex sum of linear connections $t_1 \nabla_1 + \dots t_p \nabla_p$, $t_1 + \dots + t_p = 1$; the connections are linear connections on \mathbf{M} . It is well known that $t_1 \nabla_1 + \dots t_p \nabla_p$ is also a linear connection. Now, consider the manifold $\Sigma_x \subset \pi^{-1}(x) \subset \mathbf{N}$ and a set of connections on \mathbf{M} , all of them labelled by points on Σ , so there is a map $\Theta : \mathbf{M} \longrightarrow (\mathbf{R}^+)$ such that $\int_{\Sigma_x} \Theta = 1$ and that $\Theta \geq 0$. Then, one can use a limit argument (work in progress) to show that the averaged of the family of connections $\{\nabla_u\}$ defines also a linear connection on \mathbf{M} . To apply to our case this argument, we only need to specify that $\Sigma_x = \mathbf{I}_x$ and that $\Theta(u) = d\text{vol } \pi_2|_u \nabla_{\iota_u} \pi^*$, where the right hand side must be understood for fixed $u \in \mathbf{I}_x$ and as acting on sections of $\Gamma\mathbf{M}$. \square

The averaged covariant derivative commutes with contractions:

$$\tilde{\nabla}_X [\alpha(Z)] = \langle \pi_2|_u \nabla_{\iota_u(X)} \pi^*(\alpha(Z)) \rangle_u := \iota_{\tilde{\nabla}_X(Z)} \alpha + \iota_Z \tilde{\nabla}_X \alpha.$$

The extension of the covariant derivative $\tilde{\nabla}_X$ acting on sections of $\Gamma^{(p,q)}\mathbf{M}$ is performed in the usual way,

$$\tilde{\nabla}_X K(X_1, \dots, X_s, \alpha^1, \dots, \alpha^r) = \tilde{\nabla}_X K(X_1, \dots, X_s, \alpha^1, \dots, \alpha^r) -$$

$$-\sum_{i=1}^s K(X_1, \dots, \tilde{\nabla}_X X_i, \dots, X_s, \alpha^1, \dots, \alpha^s) + \sum_{j=1}^r K(X_1, \dots, X_s, \alpha^1, \dots, \tilde{\nabla}_X \alpha^j, \dots, \alpha^r).$$

We denote the affine connection associated with the above covariant derivative by $\tilde{\nabla}$: for every section $Y \in \mathbf{TM}$, $\tilde{\nabla}Y \in \mathbf{T}_x^*\mathbf{M} \otimes \mathbf{TM}$, $x \in \mathbf{M}$ is given by the action on pairs $(X, Y) \in \mathbf{T}_x\mathbf{M} \otimes \mathbf{TM}$,

$$\tilde{\nabla}(X, Y) := \tilde{\nabla}_X Y. \quad (4.1.7)$$

Remark. From the proof of *theorem 4.1.5* one easily recognize that the result can be applied to any other linear connection defined in the bundle $\pi^*\mathbf{TM}$.

Let us calculate the torsion of the connection $\tilde{\nabla}$. Then the torsion is given for arbitrary vector fields $X, Y \in \mathbf{TM}$ by

$$\begin{aligned} T_{\tilde{\nabla}}(X, Y) &= \langle \pi_2|_u \nabla_{\iota_u(X)} \pi_w^* \rangle_u Y - \langle \pi_2|_u \nabla_{\iota_u(Y)} \pi_w^* \rangle_u X - [X, Y] = \\ &= \langle \pi_2|_u \nabla_{\iota_u(X)} \pi_u^* \rangle_u Y - \langle \pi_2|_u \nabla_{\iota_u(Y)} \pi_u^* \rangle_u X \\ &\quad - \langle \pi_2|_u \pi_u^*[X, Y] \rangle_u = \\ &= \langle \pi_2|_u (\nabla_{\iota_u(X)} \pi^* Y - \nabla_{\iota_u(Y)} \pi^* X - \pi^*[X, Y]) \rangle_u = 0, \end{aligned}$$

because the torsion-free condition of the Chern connection. Therefore,

Proposition 4.1.6 *Let (\mathbf{M}, F) be a Finsler structure with averaged connection $\tilde{\nabla}$. Then the torsion $T_{\tilde{\nabla}}$ of the average connection obtained from the Chern connection is zero.*

Let us consider the following (non-degenerate) tensors,

$$g_t = (1 - t)g + th, \quad t \in [0, 1].$$

g_t defines a Finsler structure in \mathbf{M} . The associated Chern's connection are denoted by ∇_t . In a similar as in *theorem 4.1.5* the following result is proved:

Theorem 4.1.7 *Let (\mathbf{M}, F) be a Finsler manifold and $g_t = (1 - t)g + th$, $t \in [0, 1]$. Then the operator*

$$\tilde{\nabla}_t = \frac{1}{\text{vol}(\mathbf{I}_x)} \int_{\mathbf{I}_x} \pi_2|_u \nabla \pi_v^* \quad (4.1.8)$$

is a linear connection on \mathbf{M} with zero torsion for every $t \in [0, 1]$.

4.2 Structural theorems

We consider some results obtained in [8] relating geometric objects of the averaged connection and their related averaged objects. The results can be applied to the averaged connection of any linear connection on $\pi^*\mathbf{TM}$.

If $\mathcal{E} \rightarrow \mathbf{N}$ is an arbitrary vector bundle over \mathbf{N} , for a given parallel transport τ along an arbitrary path x_t with tangent vector $\dot{x}_t \in \mathbf{T}_u\mathbf{N}$, the covariant derivative of a section S is given by the expression

$$\nabla_{\dot{x}_t} S = \lim_{\delta \rightarrow 0} \frac{1}{\delta} (\tau_t^{t+\delta} S(x_{t+\delta}) - S(x_t)).$$

Applying this formula to the Chern connection,

$$\tilde{\nabla}_X S = \langle \pi_2|_u(t) \lim_{\delta \rightarrow 0} \frac{1}{\delta} (\tau_t^{t+\delta} \pi_{u(t+\delta)}^* S(x_{t+\delta}) - \pi_{u(t)}^* S(x_t)) \rangle_{u(t)},$$

with $u(t + \delta) \in \pi^{-1}(x(t + \delta))$. Interchanging the limit and the average operation (this can be done, because both integrals are performed on the same manifold) one obtains,

$$\tilde{\nabla}_X S = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \langle \pi_2|_u (\tau_t^{t+\delta} \pi_{u(t+\delta)}^* S(x_{t+\delta}) - \pi_{u(t)}^* S(x_t)) \rangle_{u(t)}.$$

This interchange can be done because the integration is performed in \mathbf{I}_x , not depending of the limit label δ . Therefore,

$$\begin{aligned} \langle \pi_2|_u \pi_{u(t)}^* S(x_t) \rangle_{u(t)} &= \langle \pi_2|_u \pi_{u(t)}^* S^\mu(x_t) \frac{\partial}{\partial x^\mu} \rangle_{u(t)} = \\ &= S^\mu(x_t) \langle \pi_2|_u \pi_{u(t)}^* \frac{\partial}{\partial x^\mu} \rangle_{u(t)} = S^\mu \frac{\partial}{\partial x^\mu}. \end{aligned}$$

Then one can conclude that the expression

$$\langle \pi_2|_u \tau_t^{t+\delta} \pi_{u(t+\delta)}^* \rangle_u$$

plays the role of the parallel transport operation for the average connection $\tilde{\nabla}$,

Theorem 4.2.1 *Let (\mathbf{M}, F) be a Finsler structure with associated Chern's connection ∇ and with average connection $\tilde{\nabla}$. Then the parallel transport associated with $\tilde{\nabla}$ along a short path of parameter length δt is given by*

$$(\tilde{\tau}_t^{t+\delta})_{x_t} S := \langle \pi_2|_u \tau_t^{t+\delta} \pi_{u(t+\delta)}^* S(x_{t+\delta}) \rangle_{u(t)}, \quad S_x \in \Gamma_{x_t} \mathbf{M}. \quad (4.2.1)$$

Proof: It is immediate from the definition of the covariant derivative in terms of infinitesimal parallel transport; let us define the section along γ_t by

$$\tilde{\tau}S(x_{t+\delta}) = \tilde{\tau}_{t+\delta}^t S(x_t),$$

that is the parallel transported value of the section S from the point $x_{t+\delta}$ to the point x_t . Then it follows from the general definition of covariant derivative that

$$\tilde{\nabla}_X(\tilde{\tau}S) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} (\tilde{\tau}_t^{t+\delta} S(x_{t+\delta}) - S(x_t)) = 0.$$

When this condition is written in local coordinates, it is equivalent to a system of ODEs and the result is obtained from uniqueness and existence of solutions of ODEs. \square

However, in order to check the consistency of this definition, we should check that the composition rule holds:

$$\tilde{\tau}_t^{t+\delta} \circ \tilde{\tau}_{t+\delta}^{t+2\delta} = \tilde{\tau}_t^{t+2\delta}.$$

The proof is directly obtained calculating the above composition. This calculation reveals the reason to take $\tilde{\nabla}_1$ as the average connection if one wants to preserve the rule of “average parallel transport as the parallel transport of the average connection” for finite paths:

$$\begin{aligned} \tilde{\tau}_t^{t+\delta} \circ \tilde{\tau}_{t+\delta}^{t+2\delta} &= \frac{1}{\text{vol}(\mathbf{I}_{u(t)})} \frac{1}{\text{vol}(\mathbf{I}_{u(t+\delta)})} \int_{\mathbf{I}_{x_t}} \int_{\mathbf{I}_{x_{t+\delta}}} \pi_2|_{u(t)} \tau_t^{t+\delta} \pi_{u(t+\delta)}^* \pi_2|_{u(t+\delta)} \circ \\ \tau_{t+\delta}^{t+2\delta} \pi_{u(t+2\delta)}^* &= \frac{1}{\text{vol}(\mathbf{I}_{u(t)})} \frac{1}{\text{vol}(\mathbf{I}_{u(t+\delta)})} \int_{\mathbf{I}_{x_t}} \int_{\mathbf{I}_{x_{t+\delta}}} \pi_2|_{u(t)} \tau_t^{t+\delta} \circ \tau_{t+\delta}^{t+2\delta} \pi_{u(t+2\delta)}^* = \\ &= \frac{1}{\text{vol}(\mathbf{I}_{u(t)})} \int_{\mathbf{I}_{x_t}} \pi_2|_{u(t)} \tau_t^{t+2\delta} \pi_{u(t+2\delta)}^* = \tilde{\tau}_t^{t+2\delta}. \end{aligned}$$

We use a well known formula in order to express curvature endomorphisms as an infinitesimal parallel transport ([1]): denote by $\gamma_t : [0, 1] \rightarrow \mathbf{M}$ the infinitesimal parallelogram built up from the vectors $X, Y \in \mathbf{T}_x \mathbf{M}$ with lengths equal to δt constructed using parallel transport along the integral curves of $X, Y, -X, -Y$ through a short time δt and where length is measure using the Finslerian length. Then for every linear connection, the curvature endomorphisms are given by the formula

$$\Omega(X, Y) = -\frac{d\tau(\gamma_t)}{dt}|_{t=0}. \quad (4.2.2)$$

It can be written formally like

$$I + \delta t \Omega(X, Y) = -\tau_{d\tilde{\gamma}}. \quad (4.2.3)$$

Let us denote by $\tilde{\Omega} = \tilde{R} := R^{\tilde{\nabla}}$ the curvature of $\tilde{\nabla}$ and let us recall the hh-curvature of the Chern connection,

Theorem 4.2.2 *Let (\mathbf{M}, F) be a Finsler structure. Let $\iota(X_1), \iota(X_2)$ be the horizontal lifts in $\mathbf{T}_u\mathbf{N}$ of the linear independent vectors $X_1, X_2 \in \mathbf{T}_x\mathbf{M}$. Then for every section $Y \in \Gamma\mathbf{M}$,*

$$\tilde{R}_x(X_1, X_2)Y = \langle \pi_2 R_u(\iota_u(X_1), \iota_u(X_2)) \pi_u^* Y \rangle_u, \quad u \in \mathbf{I}_x \subset \pi^{-1} \subset \mathbf{N}. \quad (4.2.4)$$

Algebraic Proof. Let us assume a local frame of vector fields. Then we can write the value of the averaged curvature endomorphism as

$$\begin{aligned} \tilde{R}_u(X, Y)Z &= \langle \pi_2|_u \nabla_{\iota_u(X)} \pi_u^*|_u \cdot \langle \pi_2|_v \nabla_{\iota_v(Y)} \pi_v^*|_v Z \rangle \rangle - \\ &\quad - \langle \pi_2|_u \nabla_{\iota_u(Y)} \pi_u^*|_u \cdot \langle \pi_2|_v \nabla_{\iota_v(X)} \pi_v^*|_v Z \rangle \rangle - \\ &\quad - \langle \pi_2|_u \nabla_{\iota_u([X, Y])} \pi_u^*|_u Z \rangle, \quad X, Y, Z \in \Gamma\mathbf{TM}. \end{aligned}$$

Using the relation (4.1.1) one can reduce the above double integral to a single integral. For instance,

$$\langle \pi_2|_u \nabla_{\iota_u(X)} \pi_u^*|_u \cdot \langle \pi_2|_v \nabla_{\iota_v(Y)} \pi_v^*|_v Z \rangle \rangle = \langle \pi_2|_u \nabla_{\iota_u(X)} \nabla_{\iota_v(Y)} \pi_v^*|_u Z \rangle$$

Therefore,

$$\begin{aligned} \tilde{R}_u(X, Y)Z &= \langle \pi_2|_u \nabla_{\iota_u(X)} \nabla_{\iota_v(Y)} \pi_v^*|_u Z \rangle - \langle \pi_2|_u \nabla_{\iota_u(Y)} \nabla_{\iota_v(X)} \pi_v^*|_u Z \rangle - \\ &\quad - \langle \pi_2|_u \nabla_{\iota_u([X, Y])} \pi_u^*|_u Z \rangle = \langle \pi_2|_u R_u([X, Y]) \pi_u^*|_u Z \rangle := \langle R \rangle_x(X, Y)Z. \end{aligned}$$

□

From this second proof one can think that given two averaged objects, if we multiply them, the product of averages is the average of the product. However this is not true, as the following counterexample shows,

$$\begin{aligned} \tilde{\nabla} \langle g \rangle &= \langle \pi_2|_u \nabla_{\iota_u X} \pi_u^* \langle \pi_2|_v g_{ij}(x, v) \pi_v^* e^i \otimes \pi_v^* e^j \rangle \rangle \neq \\ &\quad \neg \langle \pi_2|_u \nabla_{\iota_u X} g_{ij}(x, u) \pi_u^* e^i \otimes \pi_u^* e^j \rangle \rangle \end{aligned}$$

because the coefficients g_{ij} live on \mathbf{I}_x and not on \mathbf{M} .

Chapter 5

Some Applications of the Averaged Connection

In this *chapter* we present some applications of the averaged connection.

5.1 Metric compatibility of the Averaged Connection

It is interesting to consider when the averaged connection obtained from the Chern connection is Riemann metrizable, that means, when exists a Riemannian metric \tilde{h} such that $\tilde{\nabla} = \nabla^{\tilde{h}}$. The basic result is the following

Proposition 5.1.1 *Let (M, F) be a Finsler structure. Then the averaged connection $\tilde{\nabla}$ of the Chern connection ∇ is a metric irreducible connection iff the Holonomy group $Hol(\tilde{\nabla})$ is a Berger group.*

Proof: Suppose that $\tilde{\nabla}$ is metrizable. Then there is a Riemannian metric such that $\tilde{\nabla} = \nabla^{\tilde{h}}$, that is, the Levi-Civita connection of \tilde{h} . Since the torsion $T_{\tilde{\nabla}} = 0$, it implies $\tilde{\nabla}$ is a Riemannian connection and therefore in the case of irreducible metrics, the holonomy group $\tilde{\nabla}$ is a Berger group.

Conversely, let us suppose that $Hol(\tilde{\nabla})$ is an irreducible Berger group. Then it is compact. Then we can define the operation:

$$\int_{Hol(\tilde{\nabla})} d\tau; \quad \tau \in Hol(\tilde{\nabla}).$$

$d\tau$ is an invariant Haar measure of the Berger group $Hol(\tilde{\nabla})$. In particular we can use the Szabo's construction in [6] to define the following scalar

product on $\mathbf{T}_x\mathbf{M}$:

$$\tilde{h}_x(X, Y) = \int_{Hol(\tilde{\nabla})} (\tau^*X, \tau^*Y)^* d\tau; \quad X, Y \in \mathbf{T}_x\mathbf{M}. \quad (5.1.1)$$

$(,)^*$ is an arbitrary scalar product on $\mathbf{T}_x\mathbf{M}$. One extends this scalar product to the whole manifold using the holonomy group, defining a Riemannian metric \tilde{h} that is conserved by $\tilde{\nabla}$. \square

5.2 Geodesic Equivalence Problem

In order to clarify the relation between h and \tilde{h} , we use the notion of geodesic rigidity to obtain a partial answer to this question.

Definition 5.2.1 *Two Riemannian metrics h and \tilde{h} living on the manifold \mathbf{M} with $\dim(\mathbf{M}) \geq 2$ are geodesically equivalent if their sets of unparameterized geodesics coincide. The manifold \mathbf{M} is called geodesically rigid if every two geodesically equivalent metrics are proportional.*

Corollary 5.2.2 *Under the above hypothesis than before, h and \tilde{h} have the same Levi-Civita connection. Therefore h and \tilde{h} are geodesically equivalent.*

Proof: By definition h is the Levi-Civita connection of $\tilde{\nabla}$. On the other hand,

$$(\tilde{\nabla}_Z(\tilde{h}))(X, Y) = 0, \quad \forall X, Y, Z \in \Gamma\mathbf{TM}.$$

because it has been extended using the holonomy group. Therefore because $\tilde{\nabla}$ is also torsion free, it is the Levi-Civita connection of \tilde{h} . \square

Remark. The above *corollary* is stronger than geodesically equivalence condition between two metrics, because the connection is already determine.

Corollary 5.2.3 *Let (\mathbf{M}, F) be a Finsler structure such that \mathbf{M} is Riemannian geodesically rigid, with $Hol(\tilde{\nabla})$ a Berger group. Then $h = C\tilde{h}$.*

Matveev solved the problem of geodesically rigidity in Riemannian manifolds (see for instance ref. [12], [13] and [14]): to decide whether or not two given metrics with the same geodesics are equivalent. In particular, for hyperbolic manifolds, being Riemannian geodesically rigid, one obtains

Corollary 5.2.4 *Let (\mathbf{M}, F) be a Finsler structure such that \mathbf{M} is a closed manifold and such that \tilde{h} is an hyperbolic metric such that $Hol(\tilde{\nabla})$ is a Berger group. Then $h = C\tilde{h}$.*

For a Berwald space the Holonomy group $Hol(\nabla)$ is compact. Then the holonomy group $Hol(\tilde{\nabla})$ is also compact and is a Berger group. Therefore it is a direct consequence from a theorem of Matveev ([14]) the following

Corollary 5.2.5 *Let (\mathbf{M}, F) be a Berwald structure such that \mathbf{M} admits an hyperbolic Riemannian metric. Then $h = \tilde{h}$ and $\tilde{\nabla}h = 0$, where h and \tilde{h} are defined as before.*

In Finsler Geometry, the Finsler function is in general not reversible ($F(x, y) \neq F(x, -y)$). Therefore it has sense the notion of geodesic reversibility. Consider the following piece-wise differentiable curve $\gamma \cup \beta$ where $\gamma(s)$, $s \in [0, s_0]$ is a geodesic of the Chern connection and β is a geodesic but that start at the end of γ and has reverted the final vector of the first geodesic. Let us close with another simple curve that is simple Δ the above curve. We call it closed almost-geodesic triangle.

Definition 5.2.6 *We say that the structure (\mathbf{M}, F) is geodesically reversible if every almost-geodesic triangle is retractible. Otherwise is geodesic irreversible.*

Examples of geodesically reversible structure are Riemannian manifolds and reversible Finsler manifolds. A non-trivial example is provided by Randers structures. One obtains the following *proposition*,

Proposition 5.2.7 *Let (\mathbf{M}, F) be a Berwald structure. Then,*

1. *It is geodesically rigid if the Szabo's metric is Riemannian geodesically equivalent. In this case, there is a Riemannian metric with the same geodesics.*
2. *It is geodesically reversible.*

A similar result is obtained by Matveev (theorem 1 of [9]).

5.3 A rigidity property for Berwald Spaces

We start considering a generalization of some well known properties of linear connections over \mathbf{M} ([3], section 5.4) to linear connections defined on the bundle $\pi^*\mathbf{TM} \rightarrow \mathbf{N}$.

Given two linear connections ${}^1\nabla$ and ${}^2\nabla$ on the bundle $\pi^*\mathbf{TM} \rightarrow \mathbf{N}$, the difference operator

$$B : \Gamma\mathbf{TM} \otimes \Gamma\mathbf{TM} \rightarrow \pi^*\Gamma\mathbf{TM}$$

$$B_u(X, Y) = {}^1\nabla_{\iota_u(X)}\pi_u^*Y - {}^2\nabla_{\iota_u(X)}\pi_u^*Y,$$

$$u \in \mathbf{N}, X, Y \in \mathbf{\Gamma TM}$$

is an homomorphism that holds the Leibnitz rule on Y and it is \mathcal{F} -linear on X .

The symmetric and skew-symmetric components S and A of B are defined in the following way

$$S_u : \mathbf{\Gamma TM} \times \mathbf{\Gamma TM} \longrightarrow \pi_u^*\mathbf{TM}$$

$$S_u(X, Y) := \frac{1}{2} (B_u(X, Y) + B_u(Y, X)).$$

$$u \in \pi^{-1}(x), \quad X, Y \in \mathbf{\Gamma TM}.$$

Then, the following relation holds for arbitrary vector fields $X, Y \in \mathbf{\Gamma TM}$,

$$2S_u(X, Y) = {}^1\nabla_{\iota_u(X)}\pi_u^*Y - {}^2\nabla_{\iota_u(X)}\pi_u^*Y + ({}^1\nabla_{\iota_u(Y)}\pi_u^*X - {}^2\nabla_{\iota_u(Y)}\pi_u^*X) =$$

$$= ({}^1\nabla_{\iota_u(X)}\pi_u^*Y + {}^1\nabla_{\iota_u(Y)}\pi_u^*X) - ({}^2\nabla_{\iota_u(X)}\pi_u^*Y + {}^2\nabla_{\iota_u(Y)}\pi_u^*X).$$

The skew-symmetric part A is defined in a similar way,

$$A_u : \mathbf{\Gamma TM} \times \mathbf{\Gamma TM} \longrightarrow \pi_u^*\mathbf{TM}$$

$$A_u(X, Y) := \frac{1}{2} (B_u(X, Y) - B_u(Y, X)),$$

$$\forall u \in \pi^{-1}(x), \quad X \in \mathbf{T}_x\mathbf{M}, Y \in \mathbf{\Gamma TM}.$$

As for the torsion, one can define the symmetric and skew-symmetric parts S and A as a family of operators, because the above definitions are point-wise.

Then, the following relation holds for arbitrary vector fields $X, Y \in \mathbf{\Gamma TM}$,

$$2A_u(X, Y) = \nabla_1(\iota_u(X))\pi_u^*Y - \nabla_2(\iota_u(X))\pi_u^*Y - (\nabla_1(\iota_u(Y))\pi_u^*X - \nabla_2(\iota_u(Y))\pi_u^*X) =$$

$$= Tor_u(\nabla_1)(X, Y) - Tor_u(\nabla_2)(X, Y).$$

Since this relation holds point-wise for all $u \in \pi^{-1}(x) \in \mathbf{N}$ we can write

$$2A(X, Y) = Tor(\nabla_1)(X, Y) - Tor(\nabla_2)(X, Y). \quad (5.3.1)$$

Definition 5.3.1 Let ∇ be a linear connection on the vector bundle $\pi^*\mathbf{TM} \rightarrow \mathbf{N}$ with connection coefficients Γ_{jk}^i . The geodesics of ∇ are the parameterized curves $x : [a, b] \rightarrow \mathbf{M}$ solutions of the differential equations

$$\frac{d^2x^i}{ds^2} + \Gamma_{jk}^i(x, \frac{dx}{ds}) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad i, j, k = 1, \dots, n, \quad (5.3.2)$$

where $\Gamma_{jk}^i(x, y)$ are the connection coefficients of ∇ .

This differential equation can be written as

$$\nabla_{\iota_u(X)} \pi_u^* X = 0, \quad u = \frac{dx}{ds} \quad (5.3.3)$$

The following propositions are direct generalizations of the analogous results for affine connections ([3]).

Proposition 5.3.2 Let ${}^1\nabla$ and ${}^2\nabla$ be linear connections on the vector bundle $\pi^*\mathbf{TM} \rightarrow \mathbf{N}$. Then the following conditions are equivalent:

1. The connections ${}^1\nabla$ and ${}^2\nabla$ have the same geodesic curves on \mathbf{M} ,
2. $B_u(X, X) = 0$,
3. $S_u = 0$,
4. $B_u = A_u, \forall u \in \mathbf{N}$.

Proof. The proof follows the lines of ref. [3, pg 64-65]:

1. (**a** \Rightarrow **b**). If ${}^1\nabla$ and ${}^2\nabla$ have the same geodesics, then they have the same geodesic equations. Therefore

$${}^1\nabla_{\iota_u(X)} \pi^*(X) = 0 \Leftrightarrow {}^2\nabla_{\iota_u(X)} \pi^*(X) = 0.$$

This implies that

$$B_u(X, X) = {}^1\nabla_{\iota_u(X)} \pi^*(X) - {}^2\nabla_{\iota_u(X)} \pi^*(X) = 0.$$

2. (**b** \Rightarrow **c**). It is consequence of linearity,

$$0 = 2B_u(X + Y, X + Y) = 2S_u(X, Y).$$

3. (**c** \Rightarrow **d**). Trivially from the definition of B, S and A .

4. (**d** \Rightarrow **a**). If $B = A$, implies $S = 0$. In particular $S_u(X, X) = 0$, which implies

$${}^1\nabla_{\iota_u(X)}\pi^*(X) = {}^2\nabla_{\iota_u(X)}\pi^*(X).$$

From this relation and from existence and uniqueness of solutions, the parameterized geodesics of ${}^1\nabla$ and ${}^2\nabla$ coincide. \square

Proposition 5.3.3 *Let ${}^1\nabla$ and ${}^2\nabla$ be linear connections on the vector bundle $\pi^*\mathbf{TM} \rightarrow \mathbf{N}$ such that they have the same covariant derivative along vertical directions. Then ${}^1\nabla = {}^2\nabla$ iff they have the same parameterized geodesics and $Tor({}^1\nabla) = Tor({}^2\nabla)$.*

Proof: If ${}^1\nabla = {}^2\nabla$, then they have the same parameterized geodesics and torsion tensors. Conversely, if the geodesics are the same, the torsion is the same, then $B = 0$. Since by hypothesis both connections have the same covariant derivative in vertical directions, one has the associated covariant derivatives coincide. \square

Let us consider the pull-back bundle $\pi^*\mathbf{TM} \rightarrow \mathbf{N}$ and the tangent bundle $\mathbf{TM} \rightarrow \mathbf{M}$ endowed with a linear connection ∇ . The horizontal lift of ∇ (or pull-back connection, ([7, pg 57])) is a connection on $\pi^*\mathbf{TM} \rightarrow \mathbf{N}$ defined by the condition

$$(\pi^*\nabla)_{\iota(X)}\pi^*S = \pi^*(\nabla_X S), \quad \tilde{X} \in \mathbf{TM}. \quad (5.3.4)$$

The parameterized geodesics of both connections $\pi^*\nabla$ and ∇ are the same,

$$(\pi^*\nabla)_{\iota_u(X)}\pi_u^*X = 0 \quad \Leftrightarrow \quad \nabla_X X = 0,$$

In order to prove that, let us choose a local coordinate system on \mathbf{M} and let us write the geodesics equations in local coordinates. To do that, we need the connection coefficients of the above connections. In particular, the possibly non-zero connection coefficients in the natural coordinates induced from the local coordinate system are such that

$$\nabla_{\partial_j}\partial_k = \Gamma_{jk}^i\partial_i \Rightarrow \pi^*\nabla_{\delta_j}\pi^*\partial_k = \pi^*(\Gamma_{jk}^i\partial_i) = (\Gamma_{jk}^i\pi^*\partial_i).$$

Let us concentrate on Berwald spaces know. We can prove now the following,

Proposition 5.3.4 *Let ∇^{ch} be the Chern connection of a Finsler structure (\mathbf{M}, \mathbf{F}) , ∇^b the linear Berwald connection and $\langle \nabla^{ch} \rangle$ its averaged connection. Then the structure is Berwald iff $\pi^*\langle \nabla^{ch} \rangle = \nabla^{ch}$.*

Proof 1: If $\pi^* \langle \nabla^{ch} \rangle = \nabla^{ch}$, since the induced horizontal connection $\pi^* \langle \nabla^{ch} \rangle$ has the same coefficients that $\langle \nabla^{ch} \rangle$ and they live on \mathbf{M} , the structure (\mathbf{M}, F) is Berwald.

Let us suppose that the structure is Berwald. Then $\pi^* \langle \nabla^{ch} \rangle = \pi^* \langle 1 \rangle \nabla^{ch} = \nabla^{ch}$. This relation is checked writing the action of the average covariant derivative on arbitrary vector sections.

Proof 2: An alternative proof of is the following. We know that

$$Tor(\nabla^{ch}) = 0 \Rightarrow Tor(\langle \nabla^{ch} \rangle) = 0.$$

On the other hand, the parameterized geodesics of $\pi^* \langle \nabla^{ch} \rangle$ are the same than the geodesics of $\langle \nabla^{ch} \rangle$. But if the space is Berwald, the geodesic equation of $\langle \nabla^{ch} \rangle$ are the same than the geodesic equation of ∇^{ch} . From this fact it follows $\pi^* \langle \nabla^{ch} \rangle = \nabla^{ch}$, because both have zero torsion. If $\pi^* \langle \nabla^b \rangle = \nabla^b$, the Berwald connection lives on \mathbf{M} and therefore the structure is Berwald. \square

The following results is direct from Szabó's theorem,

Proposition 5.3.5 *Let (\mathbf{M}, F) be a Finsler structure. Then there is an affine equivalent Riemannian structure (\mathbf{M}, h) iff the structure is Berwald.*

Proof: if there is an affine equivalence Riemannian structure h such that its Levi-Civita connection ∇^h has the same parameterized geodesics as the linear Berwald connection ∇^b and both connection have also null torsion, then both connections are the same ([3], section 5.4) and since the connection coefficients ${}^h\Gamma_{ij}^i$ live in \mathbf{M} , the structure is Berwald. Conversely, if (\mathbf{M}, F) is Berwald, its Berwald connection is metrizable ([6]). \square .

Recall that for Berwald spaces $\nabla^b = \nabla^{ch}$. Then,

Proposition 5.3.6 *Let (\mathbf{M}, F) be a Berwald structure. Then any Riemannian metric h on \mathbf{M} such that $\nabla^b \pi^* h = 0$ implies that the associated Levi-Civita connection ∇^h leaves invariant the indicatrix under horizontal parallel transport.*

Proof: If the Riemannian structure h is conserved by the Berwald connection, $\nabla^b \pi^* h = 0$. This implies that $\langle \nabla^b \rangle h = 0$. In addition, $\langle \nabla^b \rangle$ is torsion free. Therefore, $\langle \nabla^b \rangle = \nabla^h$. If ∇^b leaves invariant the indicatrix, also $\pi^* \langle \nabla^b \rangle = \pi^* \nabla^h$ leaves invariant the structure. \square

There is a converse of this result,

Proposition 5.3.7 *Let (\mathbf{M}, F) be a Finsler structure. Then if there is a Riemannian metric h that leaves invariant the indicatrix under the parallel transport of $\pi^* \nabla^h$, the structure is Berwald.*

Proof: Let us consider such Riemannian metric h and the associated Levi-Civita connection ∇^h . The induced connection $\pi^*\nabla^h$ is torsion free, its connection coefficients in natural coordinates live on \mathbf{M} and the averaged connection $\langle \pi^*\nabla^h \rangle$ coincides with ∇^h , so $\pi^*\nabla^h = \pi^* \langle \pi^*\nabla^h \rangle = \nabla^b$. The last equality because $\pi^* \langle \pi^*\nabla^h \rangle$ leaves invariant the indicatrix and it is torsion-free, therefore must be the Berwald connection. Then the connection $\pi^* \langle \pi^*\nabla^h \rangle = \nabla^b$ has coefficients living on \mathbf{M} and the structure is Berwald. \square

5.4 A corollary on non-Berwaldian Spaces Landsberg

Let us consider a Riemannian metric h such that its parallel Riemannian transport leaves invariant the indicatrix of the Finsler metric F , following *proposition 5.3.7*. Therefore F is Berwald. Let us also consider the set of interpolating metrics,

$$F_t(x, y) = (1 - t)F(x, y) + t\sqrt{h(x)_{ij}y^iy^j}, \quad i, j = 1, \dots, n, \quad t \in [0, 1]$$

and their indicatrix,

$$\mathbf{I}_x(t) := \{F_t(x, y) = 1, y \in \mathbf{T}_x\mathbf{M}, x \in \mathbf{M}\}.$$

Since the metric F is Berwald, each of the above interpolating metrics defines an indicatrix which is invariant under the action of the Levi-Civita connection of h : the parallel transport along $\gamma(s) \subset \mathbf{M}$ of \mathbf{I}_x leads to the indicatrix over the final point of the path γ .

Let us that each of these indicatrix defines a submanifold of $\mathbf{T}_x\mathbf{M}$ of co-dimension 1 and that they are non-intersecting sub-manifolds. Therefore the union of indicatrix $\{\mathbf{I}_x(t), t \in [0, 1]\}$ defines a sub-manifold of $\mathbf{T}_x\mathbf{M}$ of co-dimension 0 that is invariant under the holonomy of the metric h .

Definition 5.4.1 *A Finsler structure (\mathbf{M}, F) is a Landsberg space if the hv-curvature P of the Chern's connection is such that $\dot{A}_{ijk} = P_{ijk}^n = 0$, where the vector field is defined as $e_n = \frac{y}{F(y)}$. A pure Landsberg space is such that it is Landsberg and it is not Berwald.*

This definition that we take of Landsberg space is a bit unusual, although can be obtained from the standard characterizations straightforwardly. In particular, the standard definition of Landsberg space is such that ([1, section 3.4])

$$0 = \dot{A}_{ikl} = -l^j P_{jikl} = \tilde{l}_j P_{ikl}^j := P_{ikl}^n.$$

Theorem 5.4.2 *Let (\mathbf{M}, F) be a Finsler space and suppose that the averaged connection $\langle \nabla^{ch} \rangle$ does not leave invariant any compact submanifolds $\mathbf{I}_x(t) \subset \mathbf{T}_x \mathbf{M}$ of codimension zero. Then the structure (\mathbf{M}, F) is a pure Landsberg space.*

Proof: suppose that the Landsberg space is Berwald. Then we know from a theorem of Szabo that this linear Berwald connection is metrizable ([6]). Then, there is a Riemannian connection ∇^h that is identified with the average connection $\langle \nabla^{ch} \rangle$ and this is in contradiction with the hypothesis of the theorem because $\pi^* \nabla^h = \pi^* \langle \nabla^{ch} \rangle = \nabla^h$ leaves invariant the set of indicatrix $\mathbf{I}_x(t)$, $\forall t \in [0, 1]$ as we show before, the union defining a submanifold of co-dimension zero of $\mathbf{T}_x \mathbf{M}$. \square

One can use *theorem 5.4.2* to argue for a strategy to solve the longest posed problem in Finsler Geometry. It is the *conjecture* that there are not pure Landsberg spaces. The idea is to use the classification of affine connections to show, using additional techniques and constrains, that in fact, there are no possible holonomies groups of affine connections ([4]) available.

The proof of the conjecture has been done by the author in dimension 2 using holonomy constrains but without using the result of *theorem 5.4.2*. This is because in dimension 2, the number of possible averaged holonomies for Landsberg spaces is small and one can check directly that in fact is not possible Landsberg spaces. The problem is that in higher dimensions, the number of possible holonomies grow. Therefore, the constrain of *theorem 5.4.2* could play a role.

Chapter 6

Conclusions

From our point of view, the results presented in this work reveals the power of a principle that we called *Convex Invariance* in ref. [2]: the intrinsic invariance of some geometric properties under a homotopy in the corresponding operator *moduli space* of connections having the same averaged. Convex invariance is the invariance under a convex homotopy from any of these linear connections to the average connection. The set of linear connections having the same averaged is an equivalence class, which is a strongly convex set. We say that a property is convex invariant if it is well define on each equivalence class.

We can see the results presented in this work from this perspective. One example of how the principle works is the problem of geodesic equivalence between different Finslerian structures. From the point of view of Convex Invariance, one states the following question:

Which properties of the geodesics are defined on each equivalence class?

We prove that in the category of Berwald spaces, the geodesics are the same on each equivalence class.

Another application of this point of view is the formulation of the Landsberg problem in the following way:

Is it the property of being Landsberg convex invariant?

If the answer is yes, there are not Landsberg spaces which are not Berwald.

Apart from the above results, the perspective adopted in the conclusion seems applicable to other properties.

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